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RIEMANN–ROCH FOR ALGEBRAIC VERSUS TOPOLOGICAL K-THEORY

R.W. THOMASON
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Let l be a prime and k a nice noetherian ring in which l is invertible. This paper extends the work of Dwyer and Friedlander by defining a mod l^ν topological K-homology theory for schemes quasiprojective over k, $G/l_*^{\nu\mathrm{Top}}(X)$. It has formal properties similar to those of the algebraic K-theory of coherent sheaves, the theory usually called K' or G theory. In particular, it is covariant for projective morphisms, and satisfies Poincaré duality in that $G/l_*^{\nu\mathrm{Top}}(X)$ is isomorphic to the toplogical K-cohomology $K/l_*^{\nu\mathrm{Top}}(X)$ if X is smooth.

There is a natural transformation from algebraic G-theory

$$\varrho(X): G/l_*^\nu(X) \to G/l_*^{\nu\mathrm{Top}}(X)$$

which extends the Dwyer–Friedlander map. The Riemann–Roch theorem says that $\varrho(X)$ is natural for projective morphisms. In particular, there are compatible long exact localization sequences for the two theories. The proof of Riemann–Roch follows the pattern set in the work of Baum, Fulton, and MacPherson [3,4], and in the work of Gillet [13,14]. For $k=\mathbb{C}$, the complex numbers, a similar theorem appears in Gillet's paper [14], and similar results were known to Baum, Fulton, and MacPherson.

Most of the work to be done lies in making a definition of $G/l_*^{\nu\mathrm{Top}}(X)$ using the etale topology. Once this is done, the usual deformation to the normal cone machinery is easily adapted to prove the theorem. This machinery also plays a critical role in showing $G/l_*^{\nu\mathrm{Top}}(X)$ is well-defined. Definitions of etale homology theories have all been rather backhanded, so these difficulties are to be expected.

In the last section I apply the results to compute the algebraic and topological K groups of some varieties with a sort of cell structure, such as reductive group schemes, homogeneous spaces, and complete rational surfaces. Howard Hiller drew my attention to these problems.

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I would like to thank Eric Friedlander for persistently demanding that I write up these results. Without Daniel Quillen's support I could not have finished this or any of my papers relating algebraic and topological K-theory.

Section 1

In this section, I establish conventions and recall some properties of algebraic and topological K-theory with supports. These properties may be found in [18] or [9] or easily deduced from results there. See also [14], §2. I will work in terms of the homotopy-theoretic spectra $K(X)$ instead of the abelian groups $K_*(X)$. These groups are just the homotopy groups of $K(X)$, so statements about homotopy equivalences between spectra or about homotopy commutative diagrams imply corresponding statements about isomorphisms between groups or about commutative diagrams of groups. This attitude is analogous to that in homological algebra of working in the derived category of chain complexes, rather than with their homology groups. See [19] for an extensive discussion.

1.1. Fix a prime power l^ν. All spectra here are to be mod l^ν spectra. Thus, $K/l^\nu(X)$ is the usual algebraic K-theory spectrum of X smashed with a mod l^ν Moore spectrum. Its homotopy groups are the mod l^ν K groups of X. Similarly, $K/l^{\nu\mathrm{Top}}(X)$ is the mod l^ν topological K-theory spectrum of X, denoted $K^{\mathrm{et}}(X;\, Z/l^\nu)$ by Dwyer–Friedlander. If $l=2$, pick $l^\nu \geq 16$; if $l=3$, pick $l^\nu \geq 9$. Then $K/l^\nu(X)$ is a homotopy associative and commutative ring spectrum, and all pairings constructed below will have the appropriate associativity properties.

The homotopy inverse limits as l^ν increases of the towers $\nu \mapsto K/l^\nu(X)$, $\nu \mapsto K/l^{\nu\mathrm{Top}}(X)$ are the l-adic spectra $K(X)\hat{_l}$, $K^{\mathrm{Top}}(X)\hat{_l}$. There are l-adic versions of all theorems below. In particular, Riemann–Roch is true for the map

$$\varrho : K(X) \to K(X)\hat{_l} \to K^{\mathrm{Top}}(X)\hat{_l}. \tag{1.1}$$

I will stick to mod l^ν formulations, however.

1.2. Let k be a regular noetherian ring of finite Krull dimension. Suppose l is invertible in k, and that all residue fields of k have bounded etale cohomological dimension. All schemes considered below are to be quasiprojective over k. I write $X \times Z$ for the fibre product over $\mathrm{Spec}(k)$.

1.3. Consider $i : X \to Y$ a closed immersion of schemes, with Y smooth and quasi-projective over k. Note Y is regular. These conditions are assumed for the rest of Section 1.

1.4. Let $K/l^\nu_X(Y)$ be homotopy equivalent to the mod l^ν spectrum associated to the Q-category of the exact category of coherent $\mathcal{O}_Y$ modules supported on X. The devisage theorem says there are canonical homotopy equivalences with the mod l^ν spectrum associated to the Q-category of the exact category of coherent $\mathcal{O}_X$ modules

$$K/l^\nu_X(Y) \simeq G/l^\nu(X). \tag{1.2}$$

There is a natural fibration sequence given by Quillen's localization theorem:

$$K/l^\nu_X(Y) \to K/l^\nu(Y) \to K/l^\nu(Y-X). \tag{1.3}$$

This sequence may in fact be used to define $K/l^\nu_X(Y)$. The construction then has the following naturality properties.

Given closed immersions $X \to X' \to Y$, there is an induced map

$$K/l^\nu_X(Y) \to K/l^\nu_{X'}(Y). \tag{1.4}$$

A map of pairs $(X, Y) \to (X', Y')$ which sends $Y-X$ into $Y'-X'$ induces a map

$$K/l^\nu_{X'}(Y') \to K/l^\nu_X(Y). \tag{1.5}$$

If $\pi : V \to Y$ is a vector bundle over Y, the 'homotopy property' of K-theory implies that the induced map (1.6) is a homotopy equivalence

$$\pi^* : K/l^\nu_X(Y) \xrightarrow{\simeq} K/l^\nu_{\pi^{-1}(X)}(V). \tag{1.6}$$

The map π^* is also a homotopy equivalence if π is a torsor under a vector bundle by [18], §7, 4.1.

There is a natural external pairing

$$K/l^\nu_{X_1}(Y_1) \wedge K/l^\nu_{X_2}(Y_2) \to K/l^\nu_{X_1 \times X_2}(Y_1 \times Y_2). \tag{1.7}$$

This induces various internal pairings. In particular, $K/l^\nu_X(Y)$ is a ring spectrum and a module spectrum over $K/l^\nu(Y)$.

1.5. Let $K/l^{\nu\mathrm{Top}}_X(Y)$ be defined as the canonical functorial homotopy fibre of the map of etale K-theory spectra [9]

$$K/l^{\nu\mathrm{Top}}_X(Y) \to K/l^{\nu\mathrm{Top}}(Y) \to K/l^{\nu\mathrm{Top}}(Y-X). \tag{1.8}$$

There is a strongly convergent Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p_X(Y; Z/l^\nu(q/2)) \Rightarrow \pi_{q-p}(K/l^{\nu\mathrm{Top}}_X(Y)). \tag{1.9}$$

Here the Tate-twisted coefficient sheaf is

$$Z/l^\nu(q/2) = \begin{cases} Z/l^\nu(i) & q = 2i, \\ 0 & q \text{ odd.} \end{cases} \tag{1.10}$$

If $\pi : V \to Y$ is a vector bundle or a torsor under a vector bundle, the homotopy property for etale cohomology implies that

$$\pi^* : K/l^{\nu\mathrm{Top}}_X(Y) \to K/l^{\nu\mathrm{Top}}_{\pi^{-1}(X)}(V) \tag{1.11}$$

is a homotopy equivalence.

There are natural external pairings analogous to those in (1.7), and similar internal pairings of the spectra for topological K-theory with supports.

Dwyer and Friedlander [9] give a natural transformation $\varrho(Y):K/l^{\nu}(Y)\to K/l^{\nu\mathrm{Top}}(Y)$. As in [10], this may be realized as a natural transformation of functors into the strict category of spectra, not just as a transformation natural up to homotopy as in [12]. Then ϱ induces a natural transformation of fibre sequences (1.3) and (1.8), and so a canonical natural map compatible with all the pairings

$$\varrho: K/l^{\nu}_X(Y)\to K/l^{\nu\mathrm{Top}}_X(Y). \tag{1.12}$$

1.6. It results easily from [20] that ϱ induces a homotopy equivalence of the localization of the algebraic K-theory spectrum by inverting the Bott element β:

$$\varrho: K/l^{\nu}_X(Y)[\beta^{-1}]\xrightarrow{\sim} K/l^{\nu\mathrm{Top}}_X(Y). \tag{1.13}$$

This is under the hypothesis that k contains primitive 16th or 9th roots of unity ζ if l is 2 or 3, respectively; more precisely, k is to be an algebra over $Z[l^{-1},\zeta]$.

If k contains a primitive lth root of unity for $l>3$, the split surjectivity of (1.13) is proved in [10]. This result may be used to simplify some proofs below.

Section 2

In this section, I use deformation to the normal cone to construct a Gysin map and verify some properties.

2.1. The conditions and notations of 1.1, 1.2, 1.3 continue to hold. Let $j:Y\to Z$ be a closed immersion of schemes smooth and quasiprojective over k.

2.2. If j is the zero section embedding of Y in a vector bundle $Z=V(E)$ over Y, a proto-Gysin map

$$\langle j_*\rangle: K/l^{\nu\mathrm{Top}}_X(Y)\to K/l^{\nu\mathrm{Top}}_X(V(E)) \tag{2.1}$$

is defined as follows. Let $\lambda_E\in\pi_0 K/l^{\nu}_Y(V(E))$ be the Thom class in algebraic K-theory, $[j_*\mathcal{O}_Y]$ ([4], p. 166). Denote also by λ_E the topological Thom class $\varrho(\lambda_E)\in\pi_0K/l^{\nu\mathrm{Top}}_Y(V(E))$. Then $\langle j_*\rangle$ is cup product with λ_E composed with the restriction map induced by $p\perp 1: V(E)\to Y\times V(E)$.

$$K/l^{\nu\mathrm{Top}}_X(Y)\xrightarrow{\cup\lambda_E} K/l^{\nu\mathrm{Top}}_{X\times Y}(Y\times V(E))\longrightarrow K/l^{\nu\mathrm{Top}}_X(V(E)). \tag{2.2}$$

2.3. Lemma. *The diagram* (2.3) *commutes up to homotopy.*

$$\begin{array}{ccc} K/I^{\vee}_X(Y) & \xrightarrow{\varrho} & K/I^{\vee\mathrm{Top}}_X(Y) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle \langle j_* \rangle} \\ K/I^{\vee}_X(Z) & \xrightarrow{\varrho} & K/I^{\vee\mathrm{Top}}_X(Z) \end{array} \tag{2.3}$$

Proof. ϱ is a natural transformation with respect to restriction maps, and respects the pairings. The lemma follows as the Gysin map j_* in algebraic K-theory is given by cup product with the Thom class (e.g. [4], p. 166).

To see this, note that j_* is the composite of the equivalences (1.2) of the two left hand members of (2.3) with $G/l^{\nu}(X)$.

Consider the diagram (2.4), where the vertical equivalences are the maps (1.2).

$$\begin{array}{ccccc} K/I^{\vee}_X(Y) & \xrightarrow{\cup \lambda_E} & K/I^{\vee}_{X\times Y}(Y \times V) & \xrightarrow{(p \perp 1)^*} & K/I^{\vee}_X(V) \\ \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\ G/l^{\nu}(X) & \longrightarrow & G/l^{\nu}(X \times Y) & \xrightarrow{(1 \perp i)^*} & G/l^{\nu}(X) \end{array} \tag{2.4}$$

The right-hand square in (2.4) commutes by the proof of [18], §7, 2.1, as $(p \perp 1)(V)$ and $X \times Y$ are Tor-independent over $Y \times V$ by a calculation similar to (2.5) below. The top horizontal map is cup product with the Thom class. To show it is j_*, it suffices to show that the bottom horizontal map is the identity on $G/l^{\nu}(X)$.

As λ_E is the class of $\mathcal{O}_Y$, the map $G/l^{\nu}(X) \to G/l^{\nu}(X \times Y)$ is induced by the functor between categories of coherent modules sending the $\mathcal{O}_X$ module $\mathcal{M}$ to $\mathcal{M} \otimes_k \mathcal{O}_Y$. I claim that $\mathcal{M} \otimes_k \mathcal{O}_Y$ and $\mathcal{O}_X = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X$ are Tor-independent over $\mathcal{O}_X \otimes_k \mathcal{O}_Y$. This is a local question, so I may restrict to an affine neighbourhood. Locally, $\mathcal{M}$ has a resolution by finitely generated projective $\mathcal{O}_X$ modules, $F_\bullet \to \mathcal{M}$. The Tors are locally the homology of the complex on the left side of (2.5),

$$(F_\bullet \otimes_k \mathcal{O}_Y) \otimes_{\mathcal{O}_X \otimes_k \mathcal{O}_Y} (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \xrightarrow{\sim} (F_\bullet \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \otimes_{\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y} (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \xrightarrow{\sim} F_\bullet \tag{2.5}$$

To see that the canonical maps in (2.5) are isomorphisms, note that this is true first if each F_n is $\mathcal{O}_X$, then if F_n is a finite sum of $\mathcal{O}_X$'s, and finally, if each F_n is a retract of such a sum. This last condition holds, as each F_n is finitely generated projective. The isomorphism (2.5) and the exactness of $F_\bullet$ show that the higher Tors vanish.

Thus $\mathcal{M} \otimes_k \mathcal{O}_Y$ is contained in the subcategory on which $(1 \perp i)^*$ which is induced by tensoring with $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X$ over $\mathcal{O}_X \otimes_k \mathcal{O}_Y$. This sends $\mathcal{M} \otimes_k \mathcal{O}_Y$ to $\mathcal{M}$ by (2.5). This shows that the bottom horizontal map is (2.4) is the identity, as required.

2.4. Returning now to the general situation of 2.1, construct the deformation to the normal cone diagram for $j: Y \to Z$ as in [22], §2 or [4], §2.

$$\begin{array}{ccccc} Y & \xrightarrow{k_1} & Y \times \mathbb{A}^1 & \xleftarrow{k_0} & Y \\ \downarrow{\scriptstyle j} & & \downarrow & & \downarrow{\scriptstyle \hat{j}} \\ Z & \xrightarrow[k_1']{} & M(Y,Z) & \xleftarrow[k_0']{} & V(N_{Y/Z}) \\ & & \downarrow{\scriptstyle \pi} & & \\ & & \mathbb{A}^1 & & \end{array} \tag{2.6}$$

Here $\hat{j}$ is the embedding of Y as the zero section of the normal bundle of the embedding j. The maps k_t are the closed immersions of Y as $Y \times \{t\}$. $\tilde{M}(Y,Z)$ is the blow up of $Z \times \mathbb{A}^1$ along $Y \times \{0\}$. $M(Y,Z)$ is the complement in $\tilde{M}(Y,Z)$ of the blow-up of $Z \times \{0\}$ along $Y \times \{0\}$. The maps k_1' and k_0' are closed immersions. The squares are cartesian and Tor-independent (see Appendix).

There are isomorphisms

$$\begin{aligned} &(\pi^{-1}(\mathbb{A}^1 - \{0\}), Y \times (\mathbb{A}^1 - \{0\})) \cong (Z, Y) \times (\mathbb{A}^1 - \{0\}), \\ &(\pi^{-1}(0), Y \times \{0\}) \cong (V(N_{Y/Z}), Y). \end{aligned} \tag{2.7}$$

2.5. Apply $K/l_X^{\mathrm{vTop}}(\ )$ to (2.6) to produce (2.8). Here $\langle j_* \rangle$ is the proto-Gysin map of 2.2.

$$\begin{array}{ccccc} K/l_X^{\mathrm{vTop}}(Y) & \xleftarrow{k_1^*} & K/l_{X \times \mathbb{A}^1}^{\mathrm{vTop}}(Y \times \mathbb{A}^1) & \xrightarrow{k_0^*} & K/l_X^{\mathrm{vTop}}(Y) \\ \downarrow{\scriptstyle j_*} & & \downarrow & & \downarrow{\scriptstyle \langle \hat{j}_* \rangle} \\ K/l_X^{\mathrm{vTop}}(Z) & \xleftarrow[k_1'^*]{} & K/l_{X \times \mathbb{A}^1}^{\mathrm{vTop}}(M(Y,Z)) & \xrightarrow[k_0'^*]{} & K/l_X^{\mathrm{vTop}}(V(N_{Y/Z})) \end{array} \tag{2.8}$$

The maps k_1^*, k_0^* are homotopy equivalences as they are inverse to a homotopy equivalence by (1.11). I claim that $k_1'^*$ and $k_0'^*$ are also homotopy equivalences, so there is a Gysin map j_*, unique up to homotopy, such that (2.8) homotopy commutes. My claim follows from Lemma 2.6.

2.6. Lemma. *Let* (2.9) *be a diagram of closed immersions, with cartesian squares, and with Y, Z, and M smooth over k.*

$$\begin{array}{ccccc} X \times \mathbb{A}^1 & \longrightarrow & Y \times \mathbb{A}^1 & \xrightarrow{\tilde{j}} & M \\ \uparrow{\scriptstyle (\mathrm{id},0)} & & \uparrow{\scriptstyle (\mathrm{id},0)} & & \uparrow{\scriptstyle k} \\ X & \longrightarrow & Y & \xrightarrow[j]{} & Z \end{array} \tag{2.9}$$

Assume $Y \times \mathbb{A}^1$ is purely of codimension d in M, and Y of codimension d in Z. Then

$$k^*: K/l^{\nu\mathrm{Top}}_{X \times \mathbb{A}^1}(M) \to K/l^{\nu\mathrm{Top}}_X(Z) \tag{2.10}$$

is a homotopy equivalence.

Proof. Consider the commutative diagram (2.11) as a diagram of functors from the category of triples $X \hookrightarrow Y \hookrightarrow Z$ to the category of spectra. Do homotopy theory in this functor category.

$$\begin{array}{ccc} K/l^{\nu\mathrm{Top}}_Y(Z) & \longrightarrow & K/l^{\nu\mathrm{Top}}_{Y-X}(Z-X) \\ \downarrow & & \downarrow \\ K/l^{\nu\mathrm{Top}}(Z) & \longrightarrow & K/l^{\nu\mathrm{Top}}(Z-X) \\ \downarrow & & \downarrow \\ K/l^{\nu\mathrm{Top}}(Z-Y) & \xrightarrow{\cong} & K/l^{\nu\mathrm{Top}}((Z-X)-(Y-X)) \end{array} \tag{2.11}$$

The columns of (2.11) are fibre sequences. By the Quetzalcoatl Lemma ([2], Lemma 1.2; in the stable homotopy version the assumption on π_1 is unnecessary) the fibres of the top two rows are homotopy equivalent. Thus, the obvious maps form natural homotopy fibre sequences

$$K/l^{\nu\mathrm{Top}}_X(Z) \to K/l^{\nu\mathrm{Top}}_Y(Z) \to K/l^{\nu\mathrm{Top}}_{Y-X}(Z-X). \tag{2.12}$$

I may assume X and Y connected. I may assume $X=Y$ is smooth. For if $X \neq Y$, $(M - X \times \mathbb{A}^1, Y \times \mathbb{A}^1 - X \times \mathbb{A}^1)$, $(Z-X, Y-X)$, $(M, Y \times \mathbb{A}^1)$, and (Z, Y) are all pairs of codimension d. By the 5-lemma applied to sequences of the form (2.12), I need only show that the maps in (2.13) are homotopy equivalences.

$$\begin{array}{ccc} K/l^{\nu\mathrm{Top}}_{Y \times \mathbb{A}^1}(M) & \xrightarrow{\sim} & K/l^{\nu\mathrm{Top}}_Y(Z), \\ K/l^{\nu\mathrm{Top}}_{(Y-X) \times \mathbb{A}^1}(M - X \times \mathbb{A}^1) & \xrightarrow{\sim} & K/l^{\nu\mathrm{Top}}_{Y-X}(Z-X). \end{array} \tag{2.13}$$

To prove (2.13), it suffices to prove Lemma 2.6 when $X=Y$ is smooth. But in this case the lemma follows from cohomological purity. For comparing the strongly converging spectral sequences (1.9), it suffices to prove that k^* is an isomorphism

$$H^p_{Y \times \mathbb{A}^1}(M; Z/l^\nu(i)) \xrightarrow{\cong} H^p_Y(Z; Z/l^\nu(i)). \tag{2.14}$$

By cohomological purity, the Leray spectral sequence computing the groups in (2.14) as cohomology of sections of the local cohomology sheaves collapses, identifying (2.14) to (2.15) (see [1], XVI 3.7, V 6.4).

$$k^*: H^{p-2d}(Y \times \mathbb{A}^1; Z/l^\nu(i-d)) \to H^{p-2d}(Y; Z/l^\nu(i-d)). \tag{2.15}$$

But this map is an isomorphism, whose inverse is the isomorphism induced by the projection $Y \times \mathbb{A}^1 \to Y$ according to the homotopy property of etale cohomology ([1], XV 2.1).

2.7. Theorem. (Preliminary Riemann–Roch for a closed immersion.) *Let $i: X \to Y$, $j: Y \to Z$ be closed immersions of schemes quasiprojective over k, with k as in* 1.2. *Suppose Y and Z are smooth over k. Assume the conditions of* 1.1. *Then ϱ is compatible with the Gysin maps: diagram* (2.16) *commutes up to homotopy.*

$$\begin{array}{ccc} K/l_X^{\nu}(Y) & \xrightarrow{\varrho} & K/l_X^{\nu\mathrm{Top}}(Y) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ K/l_X^{\nu}(Z) & \xrightarrow[\varrho]{} & K/l_X^{\nu\mathrm{Top}}(Z) \end{array} \tag{2.16}$$

Proof. Apply $\varrho: K/l^{\nu}(\ ) \to K/l^{\nu\mathrm{Top}}(\ )$ to the deformation diagram (2.6). This produces a map from diagram (2.17) to the topological version (2.8).

$$\begin{array}{ccccc} K/l_X^{\nu}(Y) & \xleftarrow{k_1^*} & K/l_{X\times\mathbb{A}^1}^{\nu}(Y\times\mathbb{A}^1) & \xrightarrow{k_0^*} & K/l_X^{\nu}(Y) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle \text{Gysin map}} & & \downarrow{\scriptstyle \hat{j}_*} \\ K/l_X^{\nu}(Z) & \xleftarrow[k_1'^*]{} & K/l_{X\times\mathbb{A}^1}^{\nu}(M(Y,Z)) & \xrightarrow[k_0'^*]{} & K/l_X^{\nu}(V(N_{Y/Z})) \end{array} \tag{2.17}$$

The squares of (2.17) commute as the squares of (2.6) are Tor-independent ([18], §7, 2.11). The horizontal maps of (2.17) are all identified to $G/l^{\nu}(X\times\mathbb{A}^1) \to G/l^{\nu}(X)$ by (1.2), and so are homotopy equivalences by the homotopy property for G-theory.

The map ϱ is natural with respect to $k_1^*, k_0^*, k_1'^*, k_0'^*$ and respects the proto-Gysin map by Lemma 2.3. The commutativity of (2.16) follows by diagram chasing.

Section 3

The notations and conventions of 1.1, 1.2, 1.3, and 2.1 remain in force. In this section I prove various functorial properties of the Gysin maps for topological K-theory. All results of Section 3 follow from the claims of the last paragraph of Section 1. A reader granting credence to these claims may proceed directly to Section 4.

3.1. Proposition. *In $X \to Y \to Z$, suppose $j: Y \to Z$ is the zero section of a vector bundle $Z = V(E)$ over Y. Then the Gysin map and proto-Gysin map of* 2.5 *and* 2.2 *agree up to homotopy*

$$\langle j_* \rangle = j_* : K/l_X^{\nu\mathrm{Top}}(Y) \to K/l_X^{\nu\mathrm{Top}}(Z). \tag{3.1}$$

Proof. This is so because the deformation diagram (2.6) in this case is isomorphic to the trivial diagram (3.2) by [22], 2.17.

$$\begin{array}{ccccc} Y & \xrightarrow{k_1} & Y\times\mathbb{A}^1 & \xleftarrow{k_0} & Y \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle j=\hat{j}} \\ V(E) & \xrightarrow[k_1']{} & V(E)\times\mathbb{A}^1 & \xleftarrow[k_0']{} & V(E) \end{array} \tag{3.2}$$

3.2. Proposition. *Let* (3.3) *be a diagram where the squares are cartesian, Y, Z, Y', Z' are smooth over k, and the right-hand square is* Tor*-independent. The maps i, j, i', j' are to be closed immersions.*

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[i]{} & Y & \xrightarrow[j]{} & Z \end{array} \tag{3.3}$$

Then the diagram (3.4) *homotopy commutes.*

$$\begin{array}{ccc} K/l_X^{\mathrm{vTop}}(Y) & \xrightarrow{j_*} & K/l_X^{\mathrm{vTop}}(Z) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p'^*} \\ K/l_{X'}^{\mathrm{vTop}}(Y') & \xrightarrow[j'_*]{} & K/l_{X'}^{\mathrm{vTop}}(Z') \end{array} \tag{3.4}$$

Proof. Consider the deformation diagram (2.6) for $X \hookrightarrow Y \hookrightarrow Z$. If one omits the $\mathbb{A}^1$, it is a diagram of schemes over $Z\times\mathbb{A}^1$. Because Z' and Y are Tor-independent over Z, the deformation diagram for $X' \hookrightarrow Y' \hookrightarrow Z'$ is the pullback of (2.6) along the map $Z'\times\mathbb{A}^1 \to Z\times\mathbb{A}^1$. This is an easy calculation using Verdier's construction in [22], §2 and the basic lemmas on Tor-independence.The case where Z' is flat over Z is done in [22].

To carry out the calculations, one must note that $Y\times\{0\}$ and $Z'\times\mathbb{A}^1$ are Tor-independent over $Z\times\mathbb{A}^1$. This follows from the hypotheses and the isomorphism

$$\mathrm{Tor}_*^{\mathcal{O}_Z[T]}(\mathcal{O}_{Z'}[T], \mathcal{O}_Y) \cong \mathrm{Tor}_*^{\mathcal{O}_Z}(\mathcal{O}_{Z'}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z[T]. \tag{3.5}$$

Considering the pull-back projection map between the deformation diagrams for $X \to Y \to Z$ and for $X' \to Y' \to Z'$, one sees that it suffices to prove (3.4) for j the zero section embedding in a vector bundle and for j_*, j'_* the proto-Gysin maps. As both are essentially cup product with the Thom class, it suffices to show that the Thom classes agree:

$$p^*\lambda_{N(Y/Z)} = \lambda_{N(Y'/Z')}. \tag{3.6}$$

But as p' and j are Tor-independent, there is an isomorphism of bundles on Y', just as when Z' is flat over Z,

$$p^*N(Y/Z) \cong N(Y'/Z'). \tag{3.7}$$

Appealing to Tor-independence yet again, $p^*\lambda_{N(Y/Z)}$ is the Thom class of $p^*N(Y/Z)$ by [18], §7, 2.11. So (3.7) implies (3.6).

3.3. Proposition. (Isotopy principle.) *Let $j: Y\times\mathbb{A}^1 \to Z\times\mathbb{A}^1$ be a closed immersion of schemes smooth over $\mathbb{A}^1$. Let $j_t: Y\times\{t\}\to Z\times\{t\}$ be the fibre over $\{t\}\in\mathbb{A}^1$. Then the Gysin maps*

$$(j_t)_*: K/l_X^{v\mathrm{Top}}(Y) \to K/l_X^{v\mathrm{Top}}(Z) \tag{3.8}$$

are all homotopic.

Proof. Consider the cartesian diagram

$$\begin{array}{ccccc} Y\times\{t\} & \longrightarrow & Z\times\{t\} & \longrightarrow & \{t\} \\ \downarrow & & \downarrow & & \downarrow \\ Y\times\mathbb{A}^1 & \longrightarrow & Z\times\mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \end{array} \tag{3.9}$$

The squares are all Tor-independent as Y and Z are flat over k and by appeal to the Appendix. Consider the diagram of topological K-theory with supports in $X\times\mathbb{A}^1$ and $X\times\{t\}$, induced by the left square of (3.9). The vertical restriction maps are homotopy equivalences, so the result follows by Proposition 3.2 and a diagram chase.

3.4. Proposition. *Let $i: X\to Y$, $j: Y\to Z$, $k: Z\to W$ be closed immersions with Y, Z, W smooth over k. Then the Gysin maps compose up to homotopy*

$$(kj)_* \cong k_* j_*: K/l_X^{v\mathrm{Top}}(Y) \to K/l_X^{v\mathrm{Top}}(W). \tag{3.10}$$

Proof. Apply $K/l_X^{v\mathrm{Top}}(\ )$ to the sides and $K/l_{X\times\mathbb{A}^1}^{v\mathrm{Top}}(\ )$ to the center column of (3.11).

$$\begin{array}{ccccc} Y & \longrightarrow & Y\times\mathbb{A}^1 & \longleftarrow & Y \\ \downarrow{\scriptstyle j} & & \downarrow & & \downarrow{\scriptstyle j} \\ Z & \longrightarrow & Z\times\mathbb{A}^1 & \longleftarrow & Z \\ \downarrow{\scriptstyle k} & & \downarrow & & \downarrow{\scriptstyle \tilde{k}} \\ W & \longrightarrow & M(W,Z) & \longleftarrow & V(N_{Z/W}) \end{array} \tag{3.11}$$

The vertical maps become Gysin maps, and the horizontal ones become restriction maps. As the squares in (3.11) are cartesian and Tor-independent, the induced diagram commutes by Proposition 3.2. The horizontal maps are homotopy equivalences, so it suffices to prove that $\hat{k}_* j_* = (\hat{k}j)_*$. Thus, I may assume k is a zero section embedding in $V(E) = W$.

Now consider (3.12), where the bottom vertical maps are zero section embeddings.

$$\begin{array}{ccccc} Y & \longrightarrow & Y \times \mathbb{A}^1 & \longleftarrow & Y \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle \hat{j}} \\ Z & \longrightarrow & M(Z, Y) & \longleftarrow & V(N_{Y/Z}) \\ {\scriptstyle \hat{k}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \hat{\hat{k}}} \\ V(E) & \longrightarrow & M(Z, Y) \times_{Z \times \mathbb{A}^1} (V(E) \times \mathbb{A}^1) & \longleftarrow & V(N_{Y/Z} \oplus E \mid Y) \end{array} \qquad (3.12)$$

The squares are again cartesian and Tor-independent, and I may reduce to showing that $\hat{\hat{k}}_* \hat{j}_* \simeq (\hat{\hat{k}}\hat{j})_*$. But these are Gysin maps for zero section embeddings. The result follows from Proposition 3.1 and the fact that the Thom class of a sum of vector bundles is the cup product of the Thom classes.

3.5. Lemma. *Let $X \to X' \to Y \xrightarrow{j} Z$ be closed immersions with Y and Z smooth over k. Then the diagram* (3.13) *of Gysin maps and canonical maps* [(1.4)] *homotopy commutes.*

$$\begin{array}{ccc} K/l_X^{v\mathrm{Top}}(Y) & \longrightarrow & K/l_{X'}^{v\mathrm{Top}}(Y) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ K/l_X^{v\mathrm{Top}}(Z) & \longrightarrow & K/l_{X'}^{v\mathrm{Top}}(Z) \end{array} \qquad (3.13)$$

Proof. Note that both X and X' are contained in Y. Apply $K/l_X^{v\mathrm{Top}}(\ ) \to K/l_{X'}^{v\mathrm{Top}}(\ )$ to the deformation diagram (2.6) to produce a map of diagrams (2.8) for X and X'. One sees that it suffices to handle the case of the proto-Gysin maps, where the assertion is clear.

3.6. Theorem. *Let $X \to Y \xrightarrow{j} Z$ be closed immersions with Y and Z smooth over k. Let the conditions of* 1.1, 1.2, 1.3, *and* 2.1 *hold as usual. Then the Gysin map*

$$j_* : K/l_X^{v\mathrm{Top}}(Y) \to K/l_X^{v\mathrm{Top}}(Z)$$

is a homotopy equivalence.

3.7. Lemma. *The assertion of* 3.6 *holds if $X = Y$.*

Proof. By deformation to the normal bundle, it suffices to do the case where $Z = V(E)$ is a vector bundle over Y, and j_* is the proto-Gysin map of the zero section embedding. I may reduce to the case where Y is connected, and E is of constant rank d.

Locally on the etale site of Y, cup product with the Thom class of E induces an isomorphism of local etale cohomology sheaves

$$\mathbb{Z}/l^\nu(i) \xrightarrow[\cong]{\cup\lambda_E} \underline{H}^{2d}_Y(V(E); \mathbb{Z}/l^\nu(i+d)). \tag{3.14}$$

Thus by cohomological purity and the local to global Leray spectral sequence, cup product with λ_E induces an isomorphism of the E_2 terms of the spectral sequences (1.9).

$$\begin{aligned} E_2^{p,q} &= H^p_Y(Y; \mathbb{Z}/l^\nu(q/2) = H^p(Y; \mathbb{Z}/l^\nu(q/2)) \\ &\xrightarrow[\cong]{\cup\lambda_E} H^{p+2d}_Y(V(E); \mathbb{Z}/l^\nu((q+2d)/2)) \cong E_2^{p+2d,q+2d}. \end{aligned} \tag{3.15}$$

Thus, cup product with λ_E induces an isomorphism

$$\pi_* K/l^{\nu\mathrm{Top}}_Y(Y) \cong \pi_* K/l^{\nu\mathrm{Top}}_Y(V(E)). \tag{3.16}$$

This proves the lemma. Alternatively, 3.7 follows from (1.2), 2.3 and 1.6.

Proof of Theorem 3.6. Consider the homotopy commutative diagram (3.17), which has the homotopy type of a strictly commutative diagram of fibre sequences.

$$\begin{array}{ccc} K/l^{\nu\mathrm{Top}}_X(Y) & \xrightarrow{j_*} & K/l^{\nu\mathrm{Top}}_X(Z) \\ \downarrow & & \downarrow \\ K/l^{\nu\mathrm{Top}}_Y(Y) & \xrightarrow{j'_*} & K/l^{\nu\mathrm{Top}}_Y(Z) \\ \downarrow & & \downarrow \\ K/l^{\nu\mathrm{Top}}_{Y-X}(Y-X) & \xrightarrow{j''_*} & K/l^{\nu\mathrm{Top}}_{Y-X}(Z-X) \end{array} \tag{3.17}$$

The equivalent strictly commuting diagram is obtained via deformation to the normal cone on replacing Z by $V(N_{Y/Z})$ and the horizontal maps by proto-Gysin maps (2.2).

The columns in (3.17) are fibration sequences by (2.12). The Gysin maps j'_* and j''_* are homotopy equivalences by Lemmas 3.7. The 5-lemma implies that j_* is a homotopy equivalence.

Section 4

Throughout this section, the conventions and notations of 1.1, 1.2 remain in force. The results of Sections 2 and 3 allow the definition of a topological K-homology theory, with properties analogous to those of the algebraic G-theory.

The reader granting credence to the claims of 1.6 and willing to assume k contains primitive 16th or 9th roots of unity if $l=2$ or 3, respectively, may save some trouble by defining $G/l^{\nu\mathrm{Top}}(X)$ as $G/l^{\nu}(X)[\beta^{-1}]$. Then the arguments below define $f_*: G/l^{\nu\mathrm{Top}}(X) \to G/l^{\nu\mathrm{Top}}(X')$ for a projective morphism $f: X \to X'$ in such a way that it is not obvious that f_* agrees with the f_* of algebraic K-theory. The Riemann–Roch theorem asserts that they do in fact agree. This gives us two ways to compute f_*; comparison of the results yields interesting identities.

As it is, I will often omit tedious elementary arguments for the well-definedness of certain maps below when this results from the split surjectivity of (1.13) as proved in [10].

4.1. Definition. Let X be affine over k. Fix a closed immersion $i: X \to \mathbb{A}^n_k$. Define

$$G/l^{\nu\mathrm{Top}}(X) = K/l_X^{\nu\mathrm{Top}}(\mathbb{A}^n_k). \tag{4.1}$$

4.2. Lemma. *$G/l^{\nu\mathrm{Top}}(X)$ is independent of the choice of closed immersion into affine space, up to a canonical equivalence.*

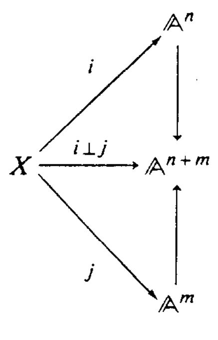
Proof. Let $i: X \to \mathbb{A}^n_k$, $j: X \to \mathbb{A}^m_k$ be two closed immersions. Let $X = \mathrm{Spec}(A)$, and let the immersions correspond to the presentations of A:

$$A \cong k[T_1, \ldots, T_n]/(p_i(T)), \qquad A \cong k[S_1, \ldots, S_m]/(q_j(S)). \tag{4.2}$$

Pick polynomials f_i, g_i such that $S_i = f_i(T_1, \ldots, T_n)$, $T_i = g_i(S_1, \ldots, S_m)$ in A. The presentation (4.3) defines the closed immersion $i \perp j: X \to \mathbb{A}^{n+m}_k$.

$$A \cong k[T_1, \ldots, T_n, S_1, \ldots, S_m]/(p_i(T), q_j(S), S_i - f_i(T), T_i - g_i(S)). \tag{4.3}$$

Consider the map of polynomial rings $k[T_1, \ldots, T_n, S_1, \ldots, S_m] \to k[T_1, \ldots, T_n]$ which sends T_i to T_i and S_i to $f_i(T)$. There is a similar map $k[T_1, \ldots, T_n, S_1, \ldots, S_m] \to k[S_1, \ldots, S_m]$. These maps fit into a commutative diagram of closed immersions



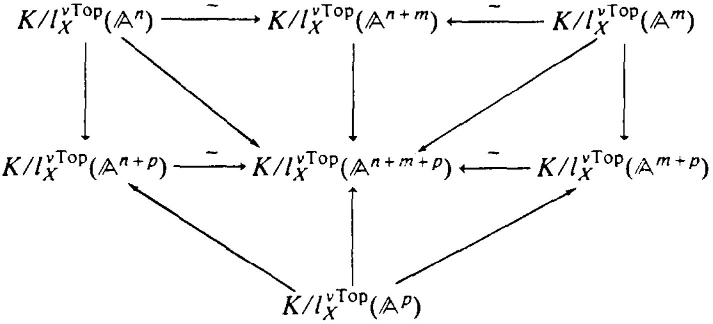
(4.4)

By Theorem 3.6, the Gysin maps are homotopy equivalences

$$K/l_X^{\nu\mathrm{Top}}(\mathbb{A}^n) \xrightarrow{\sim} K/l_X^{\nu\mathrm{Top}}(\mathbb{A}^{n+m}) \xleftarrow{\sim} K/l_X^{\nu\mathrm{Top}}(\mathbb{A}^m). \tag{4.5}$$

These Gysin equivalences do not depend of the choice of the $f_i(T)$ and $g_i(S)$. For if $f_i'(T)$ is another choice, $S_i \to \lambda f_i(T) + (1-\lambda) f_i'(T)$ defines an isotopy of immersions $\mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^{n+m} \times \mathbb{A}^1$ as λ varies over $\mathbb{A}^1$. The isotopy principle, Proposition 3.3, shows that the Gysin maps for $\lambda = 0$ and $\lambda = 1$ agree up to homotopy.

The Gysin equivalences satisfy the cocycle condition. If $k: X \to \mathbb{A}^p$ is a third closed immersion, there is a comutative diagram of Gysin equivalences



(4.6)

Thus for any choice of representative of $G/l^{\nu\mathrm{Top}}(X)$ and any other possible choice $K/l_X^{\nu\mathrm{Top}}(\mathbb{A}^n)$, there is a canonical homotopy equivalence between them which is unique up to homotopy.

4.3. If $k = \mathbb{C}$, the complex numbers, then $G/l^{\nu\mathrm{Top}}(X)$ agrees with the mod l^ν version of topological K-homology defined by Baum, Fulton, and MacPherson in [4], 3.1. This results easily from the usual comparison theorems between the etale and classical topologies. Recall that the definition in [4] of the K-homology of X is the classical $K_X^{\mathrm{Top}}(\mathbb{C}^n)$ for a topological closed embedding of X in $\mathbb{C}^n$.

4.4. Lemma. *Let $X \to W$ be a closed immersion, W smooth and affine over k. There is a canonical homotopy equivalence*

$$K/l_X^{\nu\mathrm{Top}}(W) \xrightarrow{\sim} G/l^{\nu\mathrm{Top}}(X). \tag{4.7}$$

Proof. Pick a closed immersion $W \to \mathbb{A}^n$. By Theorem 3.6, the Gysin map is a homotopy equivalence

$$K/l_X^{\nu\mathrm{Top}}(W) \xrightarrow{\sim} K/l_X^{\nu}(\mathbb{A}^n). \tag{4.8}$$

The maps (4.8) for different choices of $W \to \mathbb{A}^n$ agree under the identification of 4.2, and so define a canonical map (4.7). This map is even natural with respect to Gysin maps of closed immersions of W.

4.5. Corollary. (Affine Poincaré duality.) *Let W be smooth and affine over k. There is a canonical homotopy equivalence*

$$K/l^{\nu\mathrm{Top}}(W) = K/l_W^{\nu\mathrm{Top}}(W) \xrightarrow{\sim} G/l^{\nu\mathrm{Top}}(W). \tag{4.9}$$

4.6. Theorem. (Affine Riemann–Roch.) *Let X be affine over k. Assume* 1.1, 1.2 *hold. Then there is a canonical map*

$$\varrho : G/l^{\mathrm{v}}(X) \to G/l^{\mathrm{vTop}}(X). \tag{4.10}$$

For $X \to W$ a closed immersion with W affine and smooth over k, ϱ is compatible with the ϱ of (1.12) *under the maps of* (1.2) *and* (4.7) *in that* (4.11) *is homotopy commutative.*

$$\begin{array}{ccc} G/l^{\mathrm{v}}(X) & \xrightarrow{\ \varrho\ } & G/l^{\mathrm{vTop}}(X) \\ \uparrow & & \uparrow \\ K/l^{\mathrm{v}}_X(W) & \xrightarrow{\ \varrho\ } & K/l^{\mathrm{vTop}}_X(W) \end{array} \tag{4.11}$$

Proof. Pick a closed immersion $X \to \mathbb{A}^n$ and use (4.11) with $W = \mathbb{A}^n$ to define the ϱ of (4.10). The ϱ's for different immersions $X \to W$ agree under the Gysin map equivalences by the preliminary Riemann–Roch theorem, 2.7.

4.7. Proposition. *Let X be affine over k, $p : \tilde{X} \to X$ a torsor under a vector bundle E on X. Then there is a homotopy equivalence*

$$p^* : G/l^{\mathrm{vTop}}(X) \xrightarrow{\ \simeq\ } G/l^{\mathrm{vTop}}(\tilde{X}). \tag{4.12}$$

If $p' : \tilde{X}' \to X$ is a torsor under a vector bundle E', the projection $q : \tilde{X} \times_X \tilde{X}' \to \tilde{X}$ is a torsor under p^*E, and diagram (4.13) *commutes up to homotopy.*

$$\begin{array}{ccc} G/l^{\mathrm{vTop}}(\tilde{X} \times_X \tilde{X}') & & \\ \uparrow{\scriptstyle (p\times p')^*} & \nwarrow{\scriptstyle q^*} & \\ & & G/l^{\mathrm{vTop}}(\tilde{X}) \\ & \nearrow{\scriptstyle p^*} & \\ G/l^{\mathrm{vTop}}(X) & & \end{array} \tag{4.13}$$

Proof. Torsors under E on X are classified by the Zariski cohomology group $H^1(X; E)$; as E is a coherent sheaf and X is affine, this group vanishes. Thus the torsor $\tilde{X}$ has a section and is isomorphic to the vector bundle $V(E)$ over X. There is a Grassmanian Y over k with its canonical vector bundle $\mathscr{E}$ and a map $f : X \to Y$ such that $E = f^*\mathscr{E}$. Let $\pi : \tilde{Y} \to Y$ be an affine resolution of Y, as in [17], 1.5. $\tilde{Y}$ is affine and is a torsor under a vector bundle over Y. Then $f^*\tilde{Y}$ is a torsor on X, and so has a section. Thus f lifts to a map $\tilde{f} : X \to \tilde{Y}$.

Pick a closed immersion $i : X \to \mathbb{A}^n_k$. Then $i \perp \tilde{f} : X \to \mathbb{A}^n \times \tilde{Y}$ is a closed immersion into an affine scheme smooth over k. Let E' be the pullback of $\mathscr{E}$ to $\mathbb{A}^n \times \tilde{Y}$. Then E' restricts to E on X, so $V(E')$ pulls back to $V(E) \cong \tilde{X}$ over X. Appealing to the homotopy property (1.11) and Lemma 4.4, I obtain a diagram (4.14) of homotopy equivalences which defines p^*,

$$\begin{array}{ccc} K/I^{\mathrm{vTop}}_{V(E)}(V(E')) & \xrightarrow{\sim} & G/I^{\mathrm{vTop}}(V(E)) \\ \uparrow & & \uparrow p^* \\ K/I^{\mathrm{vTop}}_{X}(\mathbb{A}^n\times\tilde{Y}) & \xrightarrow{\sim} & G/I^{\mathrm{vTop}}(X) \end{array} \tag{4.14}$$

That p^* is independent of the choices made follows from the isotopy principle 3.3 and 4.6, 4.2, 3.2. Note that the choices of f yield isotopic immersions $i\perp f$ by [11], Lemma 3.3. The independence of p^* also follows from the claims of 1.6 and Lemma 4.8.

The functorial property (4.13) follows similarly from Lemma 4.8 and 1.6, or by an easy elementary argument.

4.8. Lemma. *Let $p:\tilde{X}\to X$ be a torsor under a vector bundle with X affine over k. Then diagram* (4.15) *homotopy commutes, where the left p^* is the map in G-theory induced by the flat map p.*

$$\begin{array}{ccc} G/I^{\mathrm{v}}(\tilde{X}) & \xrightarrow{\varrho} & G/I^{\mathrm{vTop}}(\tilde{X}) \\ p^*\uparrow & & \uparrow p^* \\ G/I^{\mathrm{v}}(X) & \xrightarrow{\varrho} & G/I^{\mathrm{vTop}}(X) \end{array} \tag{4.15}$$

Proof. Consider the analogue of (4.14) for algebraic G-theory. This analogue commutes by [18], §7, Prop. 2.11. The map ϱ maps this analogue to (4.14), forming a cubical diagram with (4.14) as the back face and the analogue as the front face. The left face of the cube commutes by the usual naturality of ϱ. The top and bottom faces commute by the preliminary Riemann–Roch theorem of 2.7. The horizontal maps on the top and bottom faces are homotopy equivalences. A diagram chase shows that the right face of the cube commutes. But this is precisely (4.15).

4.9. Definition. Let X be quasiprojective over k. Choose an affine resolution $p:\tilde{X}\to X$ as in [17], 1.5. Such an $\tilde{X}$ is affine over k, and p makes $\tilde{X}$ a torsor under a vector bundle over X. Define

$$G/I^{\mathrm{vTop}}(X)=G/I^{\mathrm{vTop}}(\tilde{X}). \tag{4.16}$$

Define a map $\varrho(X)$ up to homotopy

$$\varrho(X):G/I^{\mathrm{v}}(X)\to G/I^{\mathrm{vTop}}(X) \tag{4.17}$$

by $\varrho(X)=\varrho(\tilde{X})$. Then $G/I^{\mathrm{vTop}}(X)$ and $\varrho(X)$ are up to homotopy independent of the choice of $\tilde{X}$. For if $p':\tilde{X}'\to X$ is another affine resolution. Let $\tilde{X}''$ be the fibre product of $\tilde{X}$ and $\tilde{X}'$ over X. Then $\tilde{X}''$ is a torsor under a vector bundle over $\tilde{X}$ and $\tilde{X}'$, and so an affine resolution of X. By Proposition 4.7, there are homotopy equivalences

$$G/l^{\nu\mathrm{Top}}(\tilde{X}) \to G/l^{\nu\mathrm{Top}}(\tilde{X}'') \leftarrow G/l^{\nu\mathrm{Top}}(\tilde{X}'). \tag{4.18}$$

Further, these equivalences satisfy the cocycle condition analogous to (4.6). Thus $G/l^{\nu\mathrm{Top}}(X)$ is defined up to a canonical equivalence. The $\varrho(\tilde{X})$ agree under these equivalences by Lemma 4.9, so they define a $\varrho(X)$.

4.10. Proposition. (Duality.) *Let $i: X \to W$ be a closed immersion, W smooth and quasiprojective over k. Then there is a canonical homotopy equivalence*

$$K/l_X^{\nu\mathrm{Top}}(W) \xrightarrow{\sim} G/l^{\nu\mathrm{Top}}(X). \tag{4.19}$$

In particular, if X is smooth and quasiprojective over k there is a Poincaré duality homotopy equivalence

$$K/l^{\nu\mathrm{Top}}(X) \simeq G/l^{\nu\mathrm{Top}}(X). \tag{4.20}$$

Proof. Let $\tilde{W} \to W$ be an affine resolution of W. Let $\tilde{X}$ be the pullback of $\tilde{W}$ along $X \to W$. Then $\tilde{X} \to X$ is an affine resolution. Appeal now to 4.4 and 4.5, using the homotopy property (1.11).

4.11. Definition. Let $i: X \to Y$ be a closed immersion of schemes quasiprojective over k. (I allow X and Y to be singular, convention 1.3 no longer holds.) Define a Gysin map

$$i_*: G/l^{\nu\mathrm{Top}}(X) \to G/l^{\nu\mathrm{Top}}(Y). \tag{4.21}$$

Let $\tilde{Y}$ be an affine resolution of Y. Let $\tilde{i}: \tilde{X} \to \tilde{Y}$ be the pull-back of i. Choose a closed immersion $\tilde{Y} \to \mathbb{A}^n$. Define (4.21) to be the canonical change of support map analogous to (1.4).

$$\begin{array}{ccc} K/l_{\tilde{X}}^{\nu\mathrm{Top}}(\mathbb{A}^n) & \longrightarrow & K/l_{\tilde{Y}}^{\nu\mathrm{Top}}(\mathbb{A}^n) \\ \| & & \| \\ G/l^{\nu\mathrm{Top}}(X) & & G/l^{\nu\mathrm{Top}}(Y) \end{array} \tag{4.22}$$

Have fun verifying that this is independent of the choices made.

4.12. Theorem. (Riemann–Roch for a closed immersion.) *Let $i: X \to Y$ be a closed immersion of schemes quasiprojective over k. (Both X and Y may be singular;* 1.1, 1.2 *hold.) Then diagram* (4.23) *homotopy commutes.*

$$\begin{array}{ccc} G/l^{\nu}(X) & \xrightarrow{\varrho} & G/l^{\nu\mathrm{Top}}(X) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ G/l^{\nu}(Y) & \xrightarrow{\varrho} & G/l^{\nu\mathrm{Top}}(Y) \end{array} \tag{4.23}$$

Proof. This follows immediately from Proposition 4.7, Lemma 4.8, Theorem 4.6, and the compatibility of ϱ with the change of support map (4.22). The last compatibility is proved by taking functorial homotopy fibres of the vertical maps in the strictly commutative cube (4.24) of ϱ and restriction maps.

$$\begin{array}{ccccccc}
K/l^{v}(\mathbb{A}^n) & & \xrightarrow{\quad\varrho\quad} & & K/l^{v\mathrm{Top}}(\mathbb{A}^n) & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
& & K/l^{v}(\mathbb{A}^n) & \xrightarrow{\quad\varrho\quad} & & & K/l^{v\mathrm{Top}}(\mathbb{A}^n)\\
K/l^{v}(\mathbb{A}^n-\tilde{X}) & & \longrightarrow & & K/l^{v\mathrm{Top}}(\mathbb{A}^n-\tilde{X}) & & \downarrow\\
& \searrow & \downarrow & & & \searrow & \\
& & K/l^{v}(\mathbb{A}^n-\tilde{Y}) & \xrightarrow{\quad\varrho\quad} & & & K/l^{v\mathrm{Top}}(\mathbb{A}^n-\tilde{Y})
\end{array} \tag{4.24}$$

4.13. Theorem. (Localization theorem for $G/l^{v\mathrm{Top}}$.) *Let $i:X\to Y$ be a closed immersion of schemes quasiprojective over k. (X and Y may be singular;* 1.1 *and* 1.2 *hold.) Then there is a 'localization' homotopy fibre sequence*

$$G/l^{v\mathrm{Top}}(X)\xrightarrow{\ i_*\ }G/l^{v\mathrm{Top}}(Y)\longrightarrow G/l^{v\mathrm{Top}}(Y-X). \tag{4.25}$$

This is compatible with the localization fibre sequence of algebraic G-theory in that (4.26) has the homotopy type of a strictly commutative diagram. In particular, (4.26) homotopy commutes.

$$\begin{array}{ccccc}
G/l^{v}(X) & \xrightarrow{\ i_*\ } & G/l^{v}(Y) & \longrightarrow & G/l^{v}(Y-X)\\
\downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\varrho}\\
G/l^{v\mathrm{Top}}(X) & \xrightarrow{\ i_*\ } & G/l^{v\mathrm{Top}}(Y) & \longrightarrow & G/l^{v\mathrm{Top}}(Y-X)
\end{array} \tag{4.26}$$

Proof. Pick a closed immersion $Y\to Z$, Z smooth and quasiprojective over k. The fibre sequence (4.25) is identified under the equivalence (4.19) to the fibre sequence (2.10), by the usual compatibilities. The homotopy commutativity of (4.26) follows from the obvious naturality properties of ϱ and Theorem 4.12.

In fact, the diagram (4.26) is homotopy equivalent to the strictly commutative diagram

$$\begin{array}{ccccc}
K/l^{v}_{\tilde{X}}(\tilde{Z}) & \longrightarrow & K/l^{v}_{\tilde{Y}}(\tilde{Z}) & \longrightarrow & K/l^{v}_{\tilde{Y}-\tilde{X}}(\tilde{Z}-\tilde{X})\\
{\scriptstyle\varrho}\downarrow & & {\scriptstyle\varrho}\downarrow & & {\scriptstyle\varrho}\downarrow\\
K/l^{v\mathrm{Top}}_{\tilde{X}}(\tilde{Z}) & \longrightarrow & K/l^{v\mathrm{Top}}_{\tilde{Y}}(\tilde{Z}) & \longrightarrow & K/l^{v\mathrm{Top}}_{\tilde{Y}-\tilde{X}}(\tilde{Z}-\tilde{X})
\end{array} \tag{4.27}$$

Here $\tilde{Z} \to Z$ is an affine resoution, inducing pullback resolutions $\tilde{Y} \to Y$, $\tilde{X} \to X$. The left square of (4.27) strictly commutes by naturality of ϱ with respect to change of supports, proved by considering a diagram like (4.24) with $\tilde{Z}$ replacing $\mathbb{A}^n$.

4.14. Proposition. *Let Y be quasiprojective over k. There is a natural pairing*

$$G/l^{\nu\mathrm{Top}}(Y) \wedge K/l^{\nu\mathrm{Top}}(\mathbb{P}^n_k) \to G/l^{\nu\mathrm{Top}}(\mathbb{P}^n_Y). \tag{4.28}$$

Let $1, \eta, \eta^2, \ldots, \eta^n$ be powers of the class of the bundle $\mathcal{O}(-1)$ in $\pi_0 K/l^{\nu\mathrm{Top}}(\mathbb{P}^n_k)$. Then cup product with these classes defines a homotopy equivalence

$$\prod_1^{n+1} G/l^{\nu\mathrm{Top}}(Y) \xrightarrow{\sim} G/l^{\nu\mathrm{Top}}(\mathbb{P}^n_Y). \tag{4.29}$$

Proof. For Y smooth over k, this follows from Poincaré duality 4.10 and the well-known analogue of 4.14 for $K/l^{\nu\mathrm{Top}}$. This analogue reduces by an Atiyah–Hirzebruch spectral sequence argument to the corresponding result for etale cohomology, [15], VII 2.2.4, 2.2.6. A complete proof requires some fussing with the gradings, as in [8], XVIII 1.2.

For Y singular, pick a closed immersion $Y \to W$ with W smooth and quasi-projective over k. The pairing (4.28) is identified via (4.19) to the external cup product

$$K/l^{\nu\mathrm{Top}}_Y(W) \wedge K/l^{\nu\mathrm{Top}}(\mathbb{P}^n_k) \to K/l^{\nu\mathrm{Top}}_{\mathbb{P}^n_Y}(\mathbb{P}^n_W). \tag{4.30}$$

The equivalence (4.29) results from the smooth case of (4.29) and the fibration sequence (4.25).

Alternatively, 4.14 results from 1.6 and the G-theory analogue of 4.14 in [18], §7, 4.3.

4.15. Definition. Let $f: X \to Y$ be a projective morphism of schemes quasiprojective over k. Define $f_*: G/l^{\nu\mathrm{Top}}(X) \to G/l^{\nu\mathrm{Top}}(Y)$ as follows. Choose a factorization $f = pi$ with $i: X \to \mathbb{P}^n_Y$ a closed immersion and p the canonical projection $\mathbb{P}^n_Y \to Y$. Define $f_* = p_* \cdot i_*$. Here i_* is the Gysin map of 4.11. The map $p_*: G/l^{\nu\mathrm{Top}}(\mathbb{P}^n_Y) \to G/l^{\nu\mathrm{Top}}(Y)$ is defined as projection on the first factor indexed by $1 = \eta^0$, under the equivalence (4.29).

That f_* is well-defined follows from an elementary argument, or by 1.6 and Theorem 4.16.

4.16. Theorem. (Riemann–Roch.) *Let k satisfy* 1.1, 1.2. *Let $f: X \to Y$ be a projective morphism of schemes quasiprojective over k. (Both X and Y possibly singular.) Then* (4.31) *homotopy commutes.*

$$\begin{array}{ccc} G/l^{\nu}(X) & \xrightarrow{\varrho} & G/l^{\nu\mathrm{Top}}(X) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ G/l^{\nu}(Y) & \xrightarrow{\varrho} & G/l^{\nu\mathrm{Top}}(Y) \end{array} \tag{4.31}$$

Here the left f_* is the proper direct image map.

Proof. It suffices to prove this for a closed immersion Gysin map i_* and for p_* of the canonical projection. The Gysin case was handled in 4.12. To handle the second case, by 4.14 it suffices to prove commutativity for $p:\mathbb{P}^n_k\to\mathrm{Spec}(k)$ of the diagram obtained from (4.31) by taking homotopy groups π_0. But this is an easy and standard K_0 calculation.

For in algebraic G-theory, p_* sends the class of a coherent module $\mathcal{M}$ on $\mathbb{P}^n$ to the class which is the alternating sum of Zariski cohomology sheaves,

$$[p_*\mathcal{M}]-[R^1p_*\mathcal{M}]+\cdots+(-1)^n[R^np_*\mathcal{M}].$$

For $\mathcal{M}=\mathcal{O}_{\mathbb{P}^n}$, this is the unit class $[\mathcal{O}_k]$. For $\eta^i=\mathcal{O}(-i)$, $i=1,2,\dots,n$, it gives 0 as all their cohomology sheaves vanish.

Section 5

In this section, I use the results of Section 4 to show ϱ is an isomorphism for some schemes with punctured cell decompositions.

5.1. Theorem. *Let k satisfy* 1.1, 1.2. *Let X be quasiprojective over k. Suppose X is filtered by closed subschemes*

$$\phi=X_{-1}\subseteq X_0\subseteq\cdots\subseteq X_n=X. \tag{5.1}$$

Suppose that one of (a), (b), (c) *holds.*

(a) $\varrho:\pi_pG/l^{\nu}(X_i-X_{i-1})[\beta^{-1}]\to\pi_pG/l^{\nu\mathrm{Top}}(X_i-X_{i-1})$ *is an isomorphism for $0\le i\le n$ and all p.*

(b) $\varrho:\pi_pG/l^{\nu}(X_i-X_{i-1})\to\pi_pG/l^{\nu\mathrm{Top}}(X_i-X_{i-1})$ *is an isomorphism for $0\le i\le n$ and all $p\ge 0$.*

(c) $\varrho:\pi_pG/l^{\nu}(X_i-X_{i-1})\to\pi_pG/l^{\nu\mathrm{Top}}(X_i-X_{i-1})$ *is an isomorphism for $0\le i\le n$ and all $p\ge P$.*

Then, respectively,

(a) $\varrho:\pi_pG/l^{\nu}(X)[\beta^{-1}]\to\pi_pG/l^{\nu\mathrm{Top}}(X)$ *is an isomorphism for all p.*

(b) $\varrho:\pi_pG/l^{\nu}(X)\to\pi_pG/l^{\nu\mathrm{Top}}(X)$ *is an isomorphism for all $p\ge 0$.*

(c) $\varrho:\pi_pG/l^{\nu}(X)\to\pi_pG/l^{\nu\mathrm{Top}}(X)$ *is an isomorphism for all $p\ge P+n$.*

Proof. Prove the conclusion inductively in i for $\pi_pG/l^{\nu\mathrm{Top}}(X_i)$, using the localization sequence of 4.13 and the 5-lemma. The case (a) is easy (and obsolete by 1.6). The cases (b) and (c) are harder as $\pi_p \varrho$ will not in general be an isomorphism for negative p.

To handle case (b), note that $\pi_0 G/l^\nu(X) = G_0(X) \otimes Z/l^\nu$, and that $G_0(X_i) \to G_0(X_i - X_{i-1})$ is surjective as every coherent module on $X_i - X_{i-1}$ extends over X_i. Thus $\pi_0 G/l^\nu(X_i) \to \pi_0 G/l^\nu(X_i - X_{i-1})$ is surjective. As $\pi_0 G/l^\nu(X_i - X_{i-1}) = \pi_0 G/l^{\nu\mathrm{Top}}(X_i - X_{i-1})$, it follows that $\pi_0 G/l^{\nu\mathrm{Top}}(X_i) \to \pi_0 G/l^{\nu\mathrm{Top}}(X_i - X_{i-1})$ is also surjective. These surjectivities allow one to complete the 5-lemma argument.

To handle case (c), just keep track of how much ground you lose on each induction step.

5.2. Corollary. *Let G be a reductive algebraic group over* $\bar{\mathbb{F}}_p$, $p \neq l$. *Assume* 1.1. *Then*

$$\varrho : K/l_s^\nu(G) \to K/l_s^{\nu\mathrm{Top}}(G) \tag{5.2}$$

is an isomorphism for $s \geq n + r - 1$, *where r is the rank of G and n is the maximal length of words in the Weyl group.*

Proof. Filter G by the Bruhat decomposition. Let B be a Borel subgroup and let

$$X_i = \bigcup_{l(w) \leq i} BwB \tag{5.3}$$

be the union of double cosets as w runs over the elements of the Weyl group of length at most i. Then the $X_i - X_{i-1}$ are disjoint unions of BwB's. These Schubert 'cells' are isomorphic as schemes to $U'_w \times T \times U$, where T is a maximal torus of G and U'_w, U are unipotent subgroups. The U'_w and U are affine spaces $\mathbb{A}^n$, and T is a product of r copies of $\mathscr{G}_m = \mathbb{A}^1 - \{0\}$. By Friedlander's comparison of algebraic and topological K-groups of a torus ([12], 3.4), $\varrho : K/l_s^\nu(BwB) \to K/l_s^{\nu\mathrm{Top}}(BwB)$ is an isomorphism for $s \geq r - 1$. The corollary follows now from case (c) of Theorem 5.1.

5.3. The topological K-groups $K/l_*^{\nu\mathrm{Top}}(G)$ for G reductive over $\bar{\mathbb{F}}_p$ are the same as the mod l^ν topological K-groups of the $\mathbb{C}$-form of G as a complex analytic variety. These groups are computed in [16].

5.4. Corollary. *Let X be a quasiprojective scheme over* $\bar{\mathbb{F}}_p$, $p \neq l$. *Suppose* 1.1 *holds. Suppose X has a cell decomposition, i.e. a filtration* (5.1) *by closed subschemes with each* $X_i - X_{i-1}$ *isomorphic to an affine space* $\mathbb{A}^n$. *Then there is an isomorphism for* $s \geq 0$

$$\varrho : G/l_s^\nu(X) \xrightarrow{\cong} G/l_s^{\nu\mathrm{Top}}(X). \tag{5.4}$$

If X is smooth, G may be replaced by K.

Proof. Immediate from 5.1.

5.5. In particular, (5.4) is an isomorphism if $X/\bar{\mathbb{F}}_p$ is
(1) a complete rational surface,
(2) a projective space $\mathbb{P}^n$,
(3) a flag manifold or Grassmannian,
(4) G/P, G a reductive algebraic group and P a parabolic subgroup.

Flag manifolds, Grassmanians, and $\mathbb{P}^n$ have well-known Schubert cell decompositions. They are special cases of case (4), G/P. These homogeneous spaces have a cell decomposition as in [7], 3.2, 3.3, 3.13, or [5], 5 and 6 of §1.

The classification theory of surfaces shows that any complete rational surface may be obtained from $\mathbb{P}^2$ by a sequence of blow-up and blow-downs. Filtering a surface $X = X_1$ with X_0 the point to be blown-up or the $\mathbb{P}^1$ to be blown-down and appealing to 5.1(b), one sees that whether (5.4) is an isomorphism for $s \geq 0$ is not affected by such blow-ups and blow-downs. This reduces case (1) to the special case of $\mathbb{P}^2$. To show that the hypothesis of 5.1(b) is met, consider the localization sequence resulting from (4.26) for $X_0 \to X_1$ and use the face that $G/l_{-1}^{v\mathrm{Top}}(\mathbb{P}^1) = G/l_{-1}^{v\mathrm{Top}}(\bar{\mathbb{F}}_p) = 0$.

The method of Section 5 is a direct extension of the method used by Grothendieck to calculate K_0 and by Chevalley to calculate the Chow groups of such varieties.

Appendix. Tor-independence

A.1. Two schemes X and Y over Z are said to be Tor-independent over Z if the sheaf $\mathrm{Tor}_i^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for $i > 0$. In particular, this is true if X or Y is flat over Z. Tor-independence is a local question. It is a sort of K-theoretic transversality condition, as explained in [4].

The following lemma is useful in verifying Tor-independence.

A.2. Lemma. *Let* (A.1) *be a diagram of schemes, with cartesian squares.*

$$\begin{array}{ccccc} W' & \longrightarrow & W & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & X & \longrightarrow & Z \end{array} \tag{A.1}$$

(a) *If X and Y are Tor-independent over Z, and X' and W are Tor-independent over X, then X' and Y are Tor-independent over Z.*

(b) *If X and Y are Tor-independent over Z, and X' and Y are Tor-independent over Z, then X' and W are Tor-independent over X.*

Proof. The question is local, so I may assume all schemes in (A.1) are affine.

Consider the usual spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y)) \Rightarrow \mathrm{Tor}_{p+q}^{\mathcal{O}_Z}(\mathcal{O}_{X'}, \mathcal{O}_Y). \tag{A.2}$$

By Tor-independence of X and Y over Z, it collapses, yielding an isomorphism

$$\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_W = \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Y) \cong \mathrm{Tor}_p^{\mathcal{O}_Z}(\mathcal{O}_{X'}, \mathcal{O}_Y). \tag{A.3}$$

References

[1] M. Artin, A. Grothendieck and J.-L. Verdier, Theorie des Topos et Cohomologie Etale des Schemas, Lecture Notes in Math. Nrs. 269, 270, 305 (Springer, Berlin–New York, 1972).

[2] M. Artin and B. Mazur, Etale Homotopy, Lecture Notes in Math. No. 100 (Springer, Berlin–New York, 1969).

[3] P. Baum, W. Fulton and R. MacPherson, Riemann–Roch for singular varieties, Inst. Hautes Etudes Sci. Publ. Math. 45 (1975) 101–145.

[4] P. Baum, W. Fulton and R. MacPherson, Riemann–Roch and topological K-theory for singular varieties, Acta Math. 143 (1979) 155–192.

[5] L. Bai, C. Musili and C.S. Seshadri, Cohomology of line bundles on G/B, Ann. Sci. Ec. Norm. Sup. 7 (1974) 89–139.

[6] P. Berthelot, A. Grothendieck and L. Illusie, Theorie des Intersections et Theorème de Riemann–Roch, Lecture Notes in Math. No. 225 (Springer, Berlin–New York, 1971).

[7] A. Borel and J. Tits, Complements á l'article: "Groupes rédutifs", Instit. Hautes Etudes Sci. Publ. Math. 41 (1972) 253–276.

[8] P. Delingue and N. Katz, Groupes de Monodromie en Géométrie Algébrique II, Lecture Notes in Math. No. 340 (Springer, Berlin–New York, 1973).

[9] W. Dwyer and E. Friedlander, Etale K-theory, in preparation.

[10] W. Dwyer, E. Friedlander, V. Snaith and R. Thomason, Algebraic K-theory eventually surjects onto topological K-theory, Invent. Math. 66 (1982) 481–491.

[11] E. Friedlander, Etale K-theory I: Connections with etale cohomology and algebraic vector bundles, Invent. Math. 60 (1980) 105–134.

[12] E. Friedlander, Etale K-theory II: Connections with Algebraic K-theory, Ann. Sci. Ec. Norm. Sup., to appear.

[13] H. Gillet, Riemann–Roch theorems for higher algebraic K-theory, Bull. Amer. Math. Soc. New Ser. 3 (1980) 849–852.

[14] H. Gillet, Riemann–Roch theorems for higher algebraic K-theory, Adv. in Math. 40 (1981) 203–289.

[15] A. Grothendieck, Cohomologie l-adique et Fonctions L, Lecture Notes in Math. No. 589 (Springer, Berlin–New York, 1977).

[16] L. Hodgkin, On the K-theory of Lie groups, Topology 6 (1967), 1–36.

[17] J.P. Jouanolou, Une suite exacte de Mayer–Vietoris en K-theorie algebrique, Higher K-Theories, Lecture Notes in Math. No. 341 (Springer, Berlin–New York, 1973) 293–316.

[18] D. Quillen, Higher Algebraic K-theory I, Higher K-Theories, Lecture Notes in Math. No. 341 (Springer, Berlin–New York, 1973) 85–147.

[19] R. Thomason, Algebraic K-theory and etale cohomology, Preprint (1980).

[20] R. Thomason, The Lichtenbaum–Quillen conjecture for $K/l[\beta^{-1}]$, Proc. 1981 Conference at Univ. Western Ontario, to appear.

[21] R. Thomason, Amendments to descent for $K/l[\beta^{-1}]$, in preparation.

[22] J.-L. Verdier, Le theorème de Riemann–Roch pour les intersections completes, Exposé IX in: Douady–Verdier, Seminaire de Geometrie Analytique, Astérisque 36–37 (1976).